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The Differential Invariants of Space.

By J. E. Wright.

The object of this paper is to solve the three following problems:

(I) The determination of the differential invariants of all orders, of space of any number of dimensions.

(II) The determination of invariants of any manifold in this space under any transformation which leaves its "shape" unaltered.

(III) The determination of the "deformation" invariants of any manifold in this space.

In the statement of the problems, the word "invariant" is used to include the whole class of Gaussian Invariants, Parameters and Covariants. The problems are considered solved when a method is given for determining a complete functionally independent set of invariants by direct processes.

In problem (I) for example, it will be shown that all the invariants may be expressed as the algebraic invariants of certain forms, and a method will be given for writing down these forms in succession. At one stage of the work it is found simpler to discard certain of these forms and to introduce instead a complete set of invariants due to them.

In (II) the solutions are given as the invariants of algebraic forms, with the exception of those corresponding to the set introduced in (I).

The parameters which arise in the solution of (III) are also expressed in terms of algebraic invariants, but for the determination of the Gaussian invariants, and the covariants, it is found simpler to make use of a method whereby they are expressed in a different manner. The method in question leads to the complete set of invariants in every case, but the expression of the solutions as algebraic invariants has many advantages from some points of view. The chief advantage is perhaps that the well known parameters are immediately recognized.

(III) is really equivalent to the determination of the invariants of a quadratic differential form. There is indeed an apparent limitation on the generality of the form, but it is readily seen that the generality of the results is not thereby affected, and that in fact the problem of the determination of a functionally independent complete set of invariants for a quadratic form is also solved. Lie* suggested this problem for the differential form in two variables. Using Lie's method Żorawski† investigated the invariants of the first and second orders for a general quadratic form. He also considered the question of the number of functionally independent invariants of any order. This question of number has engaged the attention of several investigators, but the complete results are given by C. N. Haskins.‡ His method does not, however, lead to the expressions for the invariants themselves. Forsyth ‖ has obtained the invariants of the first, second, and third orders, of space of three dimensions. His method is, however, unsuitable for the determination of invariants of higher orders. He has also § obtained the invariants of the first, second, and third orders which are of the type sought in (II) for a surface in space of three dimensions.

Maschke,** by the use of symbolic methods has developed a process for determining invariants of any order from known invariants. But his work does not suggest any obvious method of determining a complete set.

§1. The method pursued in this paper for the solution of problem (I) consists essentially in a factorisation of the problem into two others, each of which can be readily solved. Take any system of variables $u_1, u_2 \ldots\ldots u_m$, and let them denote a system of curvilinear coördinates in ordinary space of m dimensions. If the square of the linear element be

$$ds^2 \equiv \sum_{r=1}^{m} \sum_{s=1}^{m} a_{rs} du_r du_s ,$$

where a_{rs} is a function of the u's, it must be possible to find $x_1, x_2 \ldots\ldots x_m$, functions of the u's, such that

$$ds^2 = dx_1^2 + dx_2^2 + \ldots\ldots dx_m^2 .$$

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* Ueber Differentialinvarianten. Math. Ann., Vol. XXIV (1884), pp. 574-575.

† Ueber Biegungsinvarianten, Acta Mathematica, Vol. XVI (1892-93), pp. 1-64.

‡ Trans. Amer. Math. Soc., Vol. III (1902), pp. 71-91; also Trans. Amer. Math. Soc., Vol. V (1904), pp. 167-192.

‖ Philosophical Transactions, Series A, Vol. 202 (1903), pp. 277-333.

§ Philosophical Transactions, series A, Vol. 201 (1903), pp. 329-402.

** Trans. Amer. Math. Soc., Vol. I (1900), pp. 197-204; and Vol. IV (1903), pp. 445-469.

It is easy to prove that if one set of functions $x_1 \dots x_m$ is given, the most general possible set is given by performing a general orthogonal transformation and a translation on the set x.

In fact if $x_1, x_2 \dots x_m$ and $x'_1, x'_2 \dots x'_m$ are two different sets, there must exist the relations

$$x'_i = \beta_i + \sum_{j=1}^{m} \alpha_{ij} x_j, \qquad (i = 1, 2 \dots m)$$

where the α's and β's are constants such that

$$\sum_{k=1}^{m} \alpha_{ik} \alpha_{jk} = 0 \qquad (i \neq j)$$

$$\sum_{k=1}^{m} \alpha_{ik}^2 = 1 .$$

Now any invariant will be a function of the a's and their derivatives. It will, therefore, be a function of the x's and their derivatives. This function must be invariant under the most general orthogonal transformation and under the most general translation.

It is now easy to see that the problem considered separates into two parts:

(A) The determination of all invariants under a general transformation on the u's, and subject to the condition that the x's are invariant.

(B) The selection from these of those functions which are invariantive when a translation and an orthogonal transformation are performed on the x's.

The first of these is equivalent to the determination of all the invariants of any number of functions of the set of variables $u_1, u_2 \dots u_m$. Let these functions be $f^{(1)}(u_1, u_2 \dots u_m), f^{(2)}(u_1, u_2 \dots u_m) \dots f^{(r)}(u_1, u_2 \dots u_m)$. We take as the variables occurring in the invariants

(1) all possible derivatives of the f's.
(2) $u_1, u_2 \dots u_m$.
(3) $du_1, du_2 \dots du_m, d^2u_1, d^2u_2 \dots d^2u_m$, etc.

It is true, as pointed out by Forsyth,[*] that we may take account of the ratios of the set of variables (3), by introducing equations of the type $\phi(u_1 u_2 \dots u_m) = 0$. It is, however, simpler to preserve these variables.

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[*] Philosophical Transactions, series A, Vol. 201 (1903), p. 333.

There is one simplification which may be made ; there is no need to take account of more than m functions f, for any other function may be expressed in terms of the first m of them. Hence any derivative of this function may be expressed in terms of the functions $f^{(1)}, f^{(2)} \dots . f^{(m)}$ and their derivatives. It, therefore, follows that any invariant involving this function can be expressed in terms of $f^{(1)}, f^{(2)}, \dots . f^{(m)}$ and their derivatives. The invariant is, therefore, reducible.

To solve the problem (A), we make use of Lie's* method, with slight modifications in the details of the work. The method is the following: Let F be any invariant involving the variables specified. On this we perform the most general transformation of the group of point transformations of the variables u, and assume that it becomes F'. Then the condition for invariance is $F' = \Omega^{\mu} F$, where Ω is the Jacobian of the transformation, and μ is a number.

Let $\frac{dF}{dt}$ denote the effect on F of the most general infinitesimal operator of the group, then Ω^{μ} becomes $1 + \mu \left(\sum_{r=1}^{m} \frac{\partial \xi_r}{\partial u_r} \right) \delta t$, where $\frac{du_r}{dt} = \xi_r \, (r = 1, 2, \dots m)$,

and
$$\frac{dF}{dt} = \mu \left\{ \sum_{r=1}^{m} \frac{\partial \xi_r}{\partial u_r} \right\} F. \qquad (1)$$

If F satisfies this equation for all possible operators, then it is an invariant of the type desired. Now since the group is the most general in the variables u, the ξ's are arbitrary functions of their arguments, and hence F must be a function satisfying the system of equations obtained by equating to zero the coefficients in (1) of the various derivatives of the ξ's. As proved by Lie (l.c.) the system of equations thus obtained is complete, and therefore the number of functionally independent solutions is $M - N$, where M is the number of variables, and N the number of linearly independent equations.

To determine the equation system we require the increments under the infinitesimal transformation of the variables in F. The detailed expressions for these increments are not needed for the present work, but Forsyth's† method for their determination is preferable to Lie's, and is therefore used.

Let u_i denote the original, and u'_i the transformed variables, and let $u_i + k_i$ become $u'_i + k'_i$, $(i = 1, 2 \dots . m)$.

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*Loc. cit., pp. 564-566.

†Philosophical Transactions, series A, Vol. 201 (1903) pp. 336-340.

Then $k'_i = u'_i + k'_i - u'_i$,

$$= u_i + k_i + \xi_i\,(u + k)\,\delta t - [u_i + \xi_i\,(u)\,\delta t]$$

where $\xi_i\,(u)$ is written for $\xi_i\,(u_1\,u_2\,\ldots\,u_m)$.

Hence $k'_i = k_i + [\xi_i\,(u + k) - \xi_i\,(u)]\,\delta t$, and therefore if ϕ denotes any function of the variables u, and ϕ' its transformed,

$$\phi\,(u + k) = \phi'\,(u' + k') = \phi'\,(u' + k + \overline{\xi\,(u + k) - \xi\,(u)}\,\delta t)$$

$$= \phi'\,(u' + k) + \sum_{i=1}^{m} [\xi_i\,(u + k) - \xi_i\,(u)]\,\frac{\partial\,\phi'\,(u' + k)}{\partial\,(u'_i + k_i)}\,\delta t + \ \ldots\ldots$$ or, if small quantities of order higher than the first are neglected

$$-\frac{d\phi\,(u + k)}{dt} = \sum_{i=1}^{m} [\xi_i\,(u + k) - \xi_i\,(u)]\,\frac{\partial\,\phi\,(u + k)}{\partial\,(u_i + k_i)},$$ where $\frac{d}{dt}$ operates only on u, and not on k. This equation holds for all values of the variables k, and therefore the coefficients of corresponding powers of k on both sides may be equated. In this way are obtained the increments of all the derivatives of the functions f. For our present purpose we only require those terms in the expression for any increment which involve the highest derivatives of the ξ's.

Let $f_{\alpha_1 \alpha_2 \ldots \alpha_m}$ denote $\dfrac{\partial^n f}{\partial u_1^{\alpha_1}\,\partial u_2^{\alpha_2}\,\ldots\,\partial u_m^{\alpha_m}}$, where $\alpha_1 + \alpha_2 + \ldots + \alpha_m = n$.

Then

$$-\frac{d}{dt} f_{\alpha_1 \alpha_2 \ldots \alpha_m} = \sum_{i=1}^{m} (\xi_i)_{\alpha_1 \alpha_2 \ldots \alpha_m} \frac{\partial f}{\partial u_i} +$$ terms involving lower derivatives of the ξ's. We write $u'\ u''\ \ldots\ u^{(\lambda)}\ \ldots$ for the variables du, d^2u, $\ldots\ d^{(\lambda)}u$, $\ldots$ The increments of these variables may be written down immediately. In fact

$$\frac{d}{dt}\,u_j^{(\lambda)} = \left\{ \sum_{i=1}^{m} u'_i \frac{\partial}{\partial u_i} \right\}^{\lambda} \xi_j +$$ terms involving lower derivatives of the ξ's. The expression $\left\{ \sum u'_i \dfrac{\partial}{\partial u_i} \right\}^{\lambda}$ is supposed expanded by the multinomial theorem, and then applied as an operator to ξ_j.

It is convenient to classify invariants according to 'order'. An invariant is of the n^{th} order when the increments of its variables involve n^{th} derivatives of the ξ's but no higher derivatives. We call two invariants of the n^{th} order 'independent' when one cannot be expressed in terms of the other together with invariants of order less than n. Suppose that the invariants of orders up to $n - 1$ are known. Then the invariants of orders up to n consist of these and a certain number of independent invariants of the n^{th} order. Let F be an invariant of the n^{th} order. It must satisfy the system of equations obtained by equating to zero the coefficients of the n^{th} derivatives of the ξ's in equation (1),

$$\sum_{k=1}^{m} \frac{\partial f^{(k)}}{\partial u_i} \frac{\partial F}{\partial f^{(k)}_{\alpha_1 \alpha_2 \dots\dots \alpha_m}} + \frac{n\,!}{\alpha_1!\, \alpha_2! \dots\dots \alpha_m!} (u_1')^{\alpha_1} (u_2')^{\alpha_2} \dots\dots (u_m')^{\alpha_m} \frac{\partial F}{\partial u_i^{(n)}} = 0\,. \qquad (2)$$

where $i = 1, 2 \dots\dots m$, and $\alpha_1\ \alpha_2 \dots\dots \alpha_m$ take all positive integral and zero values subject to the condition $\alpha_1 + \alpha_2 + \dots\dots \alpha_m = n$.

In addition, F must satisfy the equations obtained from the lower derivatives of the ξ's. It is easy to see that m independent solutions of the equations (2) are $d^n f^{(1)}, d^n f^{(2)}, \dots\dots d^n f^{(m)}$, where $d^n f$ denotes the n^{th} total differential of f. Also the number of derivatives of F in these equations is obviously greater by m than the number of equations. Hence we have all the functionally independent solutions in variables of the n^{th} order, provided the equations are linearly independent. One of the determinants of the matrix of the equations may be shown to be

$$\left\{ J \begin{pmatrix} f^{(1)} & f^{(2)} & \dots\dots & f^{(m)} \\ u_1 & u_2 & \dots\dots & u_m \end{pmatrix} \right\}^{\frac{(m+r-1)\,!}{r\,!\ (m-1)\,!}}.$$

The functions f are assumed independent, and therefore their Jacobian does not vanish. Hence the determinant mentioned does not vanish, and therefore the equations are linearly independent. Hence all the solutions of the equations (2) are obtained. It may readily be verified that each of the solutions $d^n f$ satisfies the remaining equations, and is an absolute invariant; this is also obvious from the form of the solutions. We may now use these solutions to get rid of n^{th} order variables from the remaining equations. When this is done the equations become precisely those for invariants of orders less than or equal to $(n - 1)$. Continuing this process we finally arrive at the complete system of invariants,

$$d^{\lambda} f^{(\mu)}, \qquad \begin{pmatrix} \lambda = 2, 3 \dots\dots n \\ \mu = 1, 2 \dots\dots m \end{pmatrix},$$

together with the solutions of the system of equations for invariants of the first order. A slight modification is here necessary, owing to the fact that F occurs explicitly. It is easy to see that we have the m solutions $df^{(1)}\ df^{(2)}\ \dots\ df^{(m)}$, which are absolute invariants, and there yet remains one other integral, which is manifestly the Jacobian of the f's. This is a relative invariant. In fact, if the transformed variables are $U_1,\ U_2\ \dots\ U_m$,

$$J\begin{pmatrix} f^{(1)} & \dots & f^{(m)} \\ U_1 & \dots & U_m \end{pmatrix} \times J\begin{pmatrix} U_1 & \dots & U_m \\ u_1 & \dots & u_m \end{pmatrix} = J\begin{pmatrix} f^{(1)}, & f^{(2)} & \dots & f^{(m)} \\ u_1, & u_2 & \dots & u_m \end{pmatrix}$$

and therefore the μ of equation (1) is -1.

Hence we have the theorem : *A complete functionally independent system of invariants of m functions f in the variables $u_1,\ u_2,\ \dots\ u_m$, involving variables up to the n^{th} order, is given by*

$$\begin{array}{l} df^{(1)},\ df^{(2)},\ \dots\ df^{(m)}, \\ d^2f^{(1)},\ d^2f^{(2)},\ \dots\ d^2f^{(m)}, \\ \vdots \qquad\qquad\qquad \vdots \\ d^nf^{(1)},\ d^nf^{(2)},\ \dots\ d^nf^{m}, \end{array}$$

which are absolute invariants, together with

$$J\begin{pmatrix} f^{(1)}, & f^{(2)}, & \dots & f^{(m)} \\ u_1, & u_2, & \dots & u_m \end{pmatrix},$$

which is a relative invariant with $\mu = -1$.

The most general invariant is therefore a function of the f's and the above absolute invariants, multiplied by some power of J.

§2. We are now in a position to determine all the invariants of ordinary space of m dimensions.

We call the functions $f^{(1)}, f^{(2)},\ \dots\ f^{(m)},\ x_1,\ x_2,\ \dots\ x_m$. The invariants must be functions of

$$\begin{array}{cccc} & J, & & \\ x_1\ , & x_2, & \dots & x_m, \\ dx_1\ , & dx_2, & \dots & dx_m, \\ d^2x_1, & d^2x_2, & \dots & d^2x_m, \\ \vdots & & & \vdots \\ d^nx_1, & d^nx_2, & \dots & d^nx_m, \end{array}$$

and of any number of functions $f^{(k)}(x_1, x_2, \ldots\, x_m)$, and of their derivatives. They must be invariants under the most general translation, and under the most general orthogonal transformation.

As in the previous case, there is no need to consider more than m functions f, since $f^{(s)}$, $(s > m)$, can be expressed in terms of $f^{(1)}, f^{(2)} \ldots\ldots f^{(m)}$. We may at once take account of the translation; it is equivalent to the condition that the x's do not occur explicitly. For the orthogonal transformation, the transformation scheme for the x's is

$$x_i' = x_i + \left\{ \sum_{k=1}^{m} \alpha_{ik}\, x_k \right\} \delta t,$$

where $\alpha_{ik} = -\alpha_{ki}$ for all values of i and k.

We pursue the same method as before to determine the invariants. There is now, however, the important limitation that the second and higher derivatives of the increments of the x's vanish.

Let the increment of x_i be denoted by ξ_i, then $\frac{\partial \xi_i}{\partial x_i}$ is zero, and therefore the fundamental equation (1) becomes

$$\frac{dF}{dt} = 0.$$

The number of operators in $\frac{d}{dt}$ is $\frac{1}{2}\, m\, (m-1)$, since each operator corresponds to an independent constant α. Therefore, provided the operators are unconnected, the number of solutions is $\frac{1}{2}\, m\, (m-1)$ less than the number of variables. Now taking account of differentials and of differential coefficients as far as the n^{th} order, and assuming m functions f, the number of variables is

$$1 \text{ of type } J,$$

$$mn \text{ of type } d^r x_s,$$

$$m \left\{ \frac{(m+n)!}{m!\; n!} - 1 \right\} \text{ of type } \frac{\partial^\lambda f}{\partial x_1^{\beta_1}\, \partial x_2^{\beta_2} \ldots\ldots \partial x_m^{\beta_m}}.$$

Altogether there are therefore

$$\frac{(m+n)!}{(m-1)!\; n!} + (n-1)\, m + 1 \text{ variables,}$$

and therefore the number of solutions is

$$\frac{(m+n)!}{(m-1)!\,n!} + (n-1)\,m + 1 - \tfrac{1}{2}\,m\,(m-1)\,,$$

provided the equations for the invariants are independent.

In order to obtain the equations, the increments of the variables are required. For the derivatives of the f's use is made of the equation

$$-\frac{d}{dt}f\,(x+k) = \sum_{i=1}^{m}\,[\xi_i\,(x+k) - \xi_i\,(x)]\,\frac{\partial f\,(x+k)}{\partial\,(x_i+k_i)}\,.$$

In this case

$$\xi_i\,(x+k) - \xi_i\,(x) = \sum_{j=1}^{m} k_j\alpha_{ij}\,,$$

and therefore

$$-\frac{d}{dt}\,\frac{\partial^n f}{\partial x_1^{r_1},\,\partial x_2^{r_2}\,\ldots\ldots\,\partial x_m{}^{r}} = \sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_{ij}\,r_j\,\frac{\partial^n f}{\partial x_1^{s_1},\,\partial x_2^{s_2}\,\ldots\ldots\,\partial x_m{}^{s}}$$

$$\left(\begin{matrix} s_i = r_i + 1\,;\; s_j = r_j - 1\,;\; s_\lambda = r_\lambda,\; \lambda \neq i,\; \lambda \neq j\,; \\ r_1 + r_2 + \ldots\ldots r_m = n \end{matrix}\right).$$

Now consider the algebraic form of the n^{th} order in the variables $X_1, X_2, \ldots\ldots X_m$,

$$A_n \equiv \left\{\sum_{i=1}^{m} X_i\,\frac{\partial}{\partial x_i}\right\}^n f\,(x_1,\,x_2\,\ldots\ldots\,x_m)\,,$$

and let the orthogonal transformation already used be performed on the variables X. The increments of the coefficients are precisely those given above. In addition we notice that the increments for the magnitudes $d^n x$ are exactly similar to those for the magnitudes x, and that J is an absolute invariant.

We therefore immediately obtain the general result:

The functionally independent set of invariants of orders up to and including the n^{th}, of space of m dimensions, are J, and the orthogonal algebraic invariants of the system of m-ary forms

$$\begin{matrix} A_1{}^{(1)}, & A_1{}^{(2)}, & \ldots\ldots & A_1{}^{(m)}, \\ A_2{}^{(1)}, & A_2{}^{(2)}, & \ldots\ldots & A_2{}^{(m)}, \\ \vdots & & & \vdots \\ A_n{}^{(1)}, & A_n{}^{(2)}, & \ldots\ldots & A_n{}^{(m)}, \end{matrix}$$

$$\sum_{i=1}^{m} d^n x_i\,X_i, \qquad\qquad (r = 1,\,2\,\ldots\ldots\,n).$$

The most general invariant of the space under a general point transformation on the variables u, is therefore a function of the functions f, and of the algebraic invariants given above, multiplied by some power of J.

It is worthy of note that the algebraic forms A, are the polar forms of the functions J. The linear forms

$$\sum_{i=1}^{m} d^r x_i X_i$$

may be excluded, provided there are m functions f, if certain additional invariants are taken account of. In fact, it is clear that the form

$$\sum_{i=1}^{m} d^r x_i X_i$$

leads to the functionally independent set of invariants

$$\sum_{i=1}^{m} d^r x_i \frac{\partial f^{(\rho)}}{\partial x_i}, (\rho = 1, 2 \dots m)$$

and it follows at once that there is no need to retain the linear forms specified, provided we add to the set of invariants the expressions

$$d^\lambda f^{(\rho)}. \qquad \begin{pmatrix} \lambda = 1, 2 \dots n \\ \rho = 1, 2 \dots m \end{pmatrix}$$

which are obviously functionally independent invariants.

It is convenient to modify the result obtained by including the quadratic form

$$\sum_{i=1}^{m} X_i^2,$$

and then the most general invariant is seen to be a function of

$$\begin{array}{lll} df^{(1)}, & df^{(2)}, \dots & df^{(m)}, \\ d^2 f^{(1)}, & \dots & d^2 f^{(m)}, \\ \vdots & & \vdots \\ d^n f^{(1)}, & \dots & d^n f^{(m)}, \end{array}$$

and of the general algebraic invariants of the forms A, and

$$\sum_{i=1}^{m} X_i^2,$$

multiplied by some power of J.

We now transform from the variables X to new variables U given by the scheme

$$X_i = \sum_{j=1}^{m} \frac{\partial x_i}{\partial u_j} U_j, \qquad (i = 1, 2 \ldots . m).$$

The Jacobian of this transformation is J, and the discriminant of the quadratic form ΣX_i^2 changes from 1 to J^2. Hence the most general invariant is a function of the quantities $d^\lambda f$, and of the general algebraic invariants of the A's and ΣX_i^2, expressed in the new variables.

Let the form

$$\sum_{i=1}^{m} X_i^2 \text{ become } \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} U_i U_j,$$

then it will be shown that the coefficients of the algebraic forms may be expressed in terms of the a's, the f's and their derivatives with respect to the variables u. We proceed to calculate these forms in the new variables.

$$A_1 \text{ becomes } \sum_{i=1}^{m} U_i \frac{\partial f}{\partial u_i}.$$

In the general case

$$A_p^{(\rho)} = \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \cdot\cdot \sum_{\lambda=1}^{m} \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \cdot\cdot \frac{\partial x_i}{\partial u_\lambda} \frac{\partial x_j}{\partial u_\mu} \frac{\partial x_k}{\partial u_\nu} \cdot\cdot \frac{\partial^p f^{(\rho)}}{\partial x_i \partial x_j \partial x_k \cdot\cdot} U_\lambda U_\mu U_\nu \cdot\cdot,$$

where there are n letters $i, j, k \ldots .$, and also n letters $\lambda, \mu, \nu, \ldots ..$ We use

$$\frac{p!}{\lambda! \, \mu! \, \nu! \ldots .} {}_p B_{\lambda, \mu, \nu, \ldots .}$$

to denote the coefficient of U_λ, U_μ, U_ν .... in the above expression for A_p.

It is convenient to introduce at this stage the well known three index symbols of Christoffel

$$\left[\begin{matrix}\lambda, \mu \\ \nu\end{matrix}\right] \equiv \tfrac{1}{2} \left\{ \frac{\partial a_{\lambda\nu}}{\partial u_\mu} + \frac{\partial a_{\mu\nu}}{\partial u_\lambda} - \frac{\partial a_{\lambda\mu}}{\partial u_\nu} \right\}$$

and we have the relations

$$\sum_{i=1}^{m} \frac{\partial^2 x_i}{\partial u_j \, \partial u_k} \frac{\partial x_i}{\partial u_\lambda} = \left[\begin{matrix} j, k \\ \lambda \end{matrix}\right].$$

Now $\frac{\partial}{\partial u_\rho} {}_q B_{\lambda, \mu, \nu, \dots} = {}_{q+1} B_{\rho, \lambda, \mu, \nu, \dots}$

$$+ \sum_{ijk\dots}^{m} \frac{\partial^q f}{\partial x_i \, \partial x_j \, \partial x_k \dots} \cdot \left\{ \sum \frac{\partial^2 x_i}{\partial u_\rho \, \partial u_\lambda} \frac{\partial x_j}{\partial u_\mu} \frac{\partial x_k}{\partial u_\nu} \dots \right\}$$

where the second Σ denotes that there is a term corresponding to each of the letters λ, μ, ν, ... and that these are added.

Also
$$ {}_q B_{p, \mu, \nu, \dots} = \sum_{i, j, k, \dots}^{m} \frac{\partial^q f}{\partial x_i \, \partial x_j \, \partial x_k \dots} \frac{\partial x_i}{\partial u_p} \frac{\partial x_j}{\partial u_\mu} \frac{\partial x_k}{\partial u_\nu} \dots .$$

We solve the m equations obtained by giving p the values 1, 2, ... m, for the quantities

$$\sum_{j, k, \dots} \frac{\partial^q f}{\partial x_i \, \partial x_j \, \partial x_k \dots} \frac{\partial x_j}{\partial u_\mu} \frac{\partial x_k}{\partial u_\nu} \dots ,$$

and substitute the result in the above equation. Hence

$$\frac{\partial}{\partial u_\rho} {}_q B_{\lambda, \mu, \nu, \dots} = {}_{q+1} B_{\rho, \lambda, \mu, \nu, \dots}$$

$$+ \sum \left\{ \sum_{i=1}^{m} \sum_{p=1}^{m} {}_q B_{p, \mu, \nu, \dots} M_p^i \frac{\partial^2 x_i}{\partial u_\rho \, \partial u_\lambda} \right\} \frac{1}{J} ,$$

where M_p^i is the cofactor of $\frac{\partial x_i}{\partial u_p}$ in J. This equation may be written

$$ {}_{q+1} B_{\rho, \lambda, \mu, \nu, \dots} - \frac{\partial}{\partial u_\rho} {}_q B_{\lambda, \mu, \nu, \dots} =$$

$$\Sigma \frac{1}{J} \begin{vmatrix} 0 & \frac{\partial^2 x_1}{\partial u_\rho \partial u_\lambda}, & \frac{\partial^2 x_2}{\partial u_\rho \partial u_\lambda}, & \dots & \frac{\partial^2 x_m}{\partial u_\rho \partial u_\lambda} \\ {}_q B_{1, \mu, \nu, \dots} & \frac{\partial x_1}{\partial u_1}, & \frac{\partial x_2}{\partial u_1}, & \dots & \frac{\partial x_m}{\partial u_1} \\ {}_q B_{2, \mu, \nu, \dots} & \frac{\partial x_1}{\partial u_2}, & \frac{\partial x_2}{\partial u_2}, & \dots & \frac{\partial x_m}{\partial u_2} \\ \vdots & & & & \vdots \\ {}_q B_{m, \mu, \nu, \dots} & \frac{\partial u_m}{\partial x_m}, & \dots & \dots & \frac{\partial x_m}{\partial u_m} \end{vmatrix}$$

We multiply the determinant on the right by J, and observe that $J^2 = \Delta$, where Δ is the discriminant of the quadratic form

$$\sum_{r,s,} a_{rs}\, U_r\, U_s\,.$$

We thus obtain the result

$$_{q+1}B_{\rho,\lambda,\mu,\nu,\ldots\ldots} - \frac{\partial}{\partial u_\rho}\, {}_qB_{\lambda,\mu,\nu,\ldots\ldots} =$$

$$\Sigma \begin{vmatrix} 0 & , \begin{bmatrix}\rho,\lambda\\1\end{bmatrix}, & \begin{bmatrix}\rho,\lambda\\2\end{bmatrix}, & \ldots\ldots & \begin{bmatrix}\rho,\lambda\\m\end{bmatrix} \\ {}_qB_{1,\mu,\nu,\ldots\ldots} & a_{11} & a_{12} & \ldots\ldots & a_{1m} \\ {}_qB_{2,\mu,\nu,\ldots\ldots} & a_{21} & a_{22} & \ldots\ldots & a_{2m} \\ \vdots & & & & \vdots \\ {}_qB_{m,\mu,\nu,\ldots\ldots} & a_{m1} & a_{m2} & \ldots\ldots & a_{mm} \end{vmatrix} \frac{1}{\Delta}$$

Where the summation contains a term for each letter $\lambda, \mu, \nu, \ldots\ldots$.

Multiply both sides of this equation by $U_\rho\, U_\lambda\, U_\mu\, U_\nu \ldots\ldots$ and sum for all values of $\rho, \lambda, \mu, \nu, \ldots\ldots$. Also let the operator

$$\sum_{\rho=1}^{m} U_\rho \frac{\partial}{\partial u_\rho}, \text{ be denoted by } \Omega.$$

Then $S_{q+1} - \Omega\, S_q$

$$= \frac{1}{\Delta} \begin{vmatrix} 0, & F_1 & , F_2 & , \ldots\ldots & F_m \\ \frac{\partial S_q}{\partial U_1}, & a_{11} & , a_{12} & , \ldots\ldots & a_{1m} \\ \vdots & & & & \vdots \\ \frac{\partial S_q}{\partial U_m}, & a_{m1} & , a_{m2} & , \ldots\ldots & a_{mm} \end{vmatrix} \quad \ldots\ldots \quad (3)$$

Where S_q denotes the form previously called A_q, and F_p is the quadratic form

$$\sum_{\rho=1}^{m} \sum_{\lambda=1}^{m} \begin{bmatrix}\rho,\lambda\\p\end{bmatrix} U_\rho U_\lambda.$$

It thus appears that the coefficients of S_{q+1} involve the a's, their derivatives, the coefficients of the forms S_q and their derivatives. But S_1 is obviously Ωf, and therefore the coefficients of S_q involve the a's, their derivatives of orders up to $q-1$, and the derivatives of the f's of orders up to q. These forms may be readily calculated in succession by means of the given equation.

The final result may now be stated. Let $u_1, u_2 \ldots\ldots u_m$ be any coördinates in space of m dimensions, and let the element of length ds be given by the equation

$$ds^2 = \sum_{r=1}^{m} \sum_{s=1}^{m} a_{rs}\, du_r\, du_s\,.$$

A complete functionally independent system of differential invariants of orders up to and including n is given by the expressions

$$d^{\lambda} f^{(\rho)} \qquad (\lambda = 1, 2, \ldots\ldots n;\ \rho = 1,\ 2, \ldots\ldots m)$$

together with the algebraic invariants of certain m-ary forms. These forms consist of:

(I) The quadratic $\displaystyle\sum_{r=1}^{m} \sum_{s=1}^{m} a_{rs}\, U_r\, U_s\,.$

(II) n forms of orders 1, 2, ... n, corresponding to each function $f(u_1, u_2, \ldots\ldots u_m)$.

The coefficients of the forms (II) involve the derivatives of the f's, the a's and their derivatives. Equation (3) enables these forms to be written down in succession.*

§3. We next consider any manifold of dimensions r in the m dimensional space. In this case there are two types of invariants. There is, in the first place, a class of invariants corresponding to transformations which preserve the 'shape' of the manifold, that is to say do not alter distances apart of points on the manifold, the distances being measured through the m dimensional space. These contain a subclass of deformation invariants, namely functions which remain unchanged when the manifold is subjected to a transformation which merely preserves lengths measured in the manifold itself. Let the manifold in question be given by $u_{r+\lambda} = 0\ (\lambda = 1 \ldots\ldots m-r)$.

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* Forsyth, in his memoir already quoted, "*The Differential Invariants of Space*" (p. 320), gives three of the forms S, namely, those for a function ϕ when $m = 3$ and $n = 3$.

Then the first class of invariants is included in the general space set. We may now, however, take as $m - r$ of the functions f the variables $u_{r+\lambda}$. In addition $du_{r+\lambda} = 0 \;\ldots.\; d^n u_{r+\lambda} = 0$.

It is quite easy to see that the result of the last paragraph may be modified by omitting the invariants $df^{(1)} \ldots. df^{(m)}$, and taking $du_1, du_2, \ldots. du_m$, as the variables in the algebraic forms. Then the set of forms includes the $m - r$ linear ones $du_{r+\lambda}$ $(\lambda = 1, 2, \ldots. m - r)$ and therefore the invariants of the manifold are the invariants and covariants of the forms in r variables obtained by putting $du_{r+\lambda} = 0$ in the m-ary set,* and in addition there are the invariants

$$d^p f^{(\rho)} \; (p = 2, \ldots. n; \; \rho = 1, 2 \ldots. m).$$

In all the coefficients of the forms the $u_{r+\lambda}$'s $(\lambda = 1 \ldots. m - r)$ are finally put equal to zero.

There are two types of forms to be calculated, namely those belonging to a function $u_{r+\lambda}$, and those belonging to a function ϕ $(u_1, u_2, \ldots. u_r)$. We make the assumption that the variables $u_{r+\lambda}$ may be so selected that in the expression

$$ds^2 = \sum_{\alpha\beta}^{m} a_{\alpha\beta} \, du_\alpha \, du_\beta$$

the coefficient $a_{\lambda\mu} = 0$ if $\lambda > r$, $\mu \leq r$. With this selection of variables

$$S_2^{(r+\lambda)} \text{ becomes } \frac{M_\lambda}{M_0},$$

where M_i is the cofactor of b_i in

$$\begin{vmatrix} b_0 & F_{r+1}, & \cdots & F_m \\ b_1 & a_{r+1,r+1} & \cdots & a_{r+1,m} \\ \vdots & & & \vdots \\ b_{m-r} & a_{r+1,m} & \cdots & a_{m,m} \end{vmatrix}$$

The coefficients of these forms obviously cannot be expressed in terms of the magnitudes $a_{\alpha\beta}$ $(\alpha, \beta = 1, 2 \ldots. r)$. Let the coefficients in question be $L_{ij}^{(r+\lambda}$ $(ij = 1 \ldots. r)$.

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* See Grace and Young. Algebra of Invariants (1903), p. 266.

Then $S_{q+1}^{(r+\lambda)} - \Omega_1 S_q^{(r+\lambda)}$

$$= \frac{1}{\Delta_1} \begin{vmatrix} 0 & F_1 & F_2 & \ldots\ldots & F_r \\ \dfrac{\partial S_q^{(r+\lambda)}}{\partial U_1} & a_{11} & a_{12} & \ldots\ldots & a_{1r} \\ \vdots & & & & \vdots \\ \dfrac{\partial S_q^{(r+\lambda)}}{\partial U_r} & a_{r1} & & \ldots\ldots & a_{rr} \end{vmatrix}$$

where Δ_1 is the discriminant of

$$\sum_{\alpha\beta}^{r} a_{\alpha\beta} U_\alpha U_\beta, \qquad \Omega_1 \equiv \sum_{=1}^{r} U_j \frac{\partial}{\partial u_j},$$

and the variables U_j of the forms are du_j $(j = 1, \ldots\ldots r)$. Now the coefficients of the forms $F_1, F_2, \ldots\ldots F_r$ are the three index symbols $\begin{bmatrix} ij \\ k \end{bmatrix}$ $(i, j, k = 1, 2 \ldots\ldots r)$ and they are therefore expressible in terms of the derivatives of the $a_{\alpha\beta}$'s where $\alpha, \beta \ngtr r$.

It therefore follows that the coefficients of the forms $S^{(r+\lambda)}$ $(\lambda = 1, 2 \ldots m - r)$ may all be expressed in terms of the coefficients of ds^2 for the manifold, their derivatives, and the derivatives of the coefficients of the forms $S_2^{(r+\lambda)}$.

The coefficients of the forms $S_q^{(r+\lambda)}$ are seen to be the generalization of the "Fundamental Magnitudes"* of order q of a surface in space of three dimensions.

The forms corresponding to a function ϕ $(u_1, u_2, \ldots\ldots u_r)$ are exactly similar to the general forms for space, for $S_1 = \Omega_1 \phi$.

$$S_{q+1} = \Omega_1 S_q + \frac{1}{\Delta_1} \begin{vmatrix} 0 & F_1 & \ldots\ldots & F_r \\ \dfrac{\partial S_q}{\partial U_1} & a_{11} & \ldots\ldots & a_{1r} \\ \vdots & & & \vdots \\ \dfrac{\partial S_q}{\partial U_r} & a_{r1} & \ldots\ldots & a_{rr} \end{vmatrix}$$

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* See Forsyth, Messenger of Mathematics, Vol. 32, 1903, pp. 68 et seq.

We may now get rid of covariants of the forms in question by taking account of the invariants $d\phi_1 \ldots . d\phi_r$, and the final solution of problem II is obtained. The general statement is the following: Let there be any manifold in space of m dimensions, and let its linear element be given by the equation

$$ds^2 = \sum_{\alpha,\,\beta=1}^{r} a_{\alpha\beta}\, du_\alpha\, du_\beta\,.$$

In addition let there be r functions $\phi\,(u_1 \ldots . u_r)$. Then a complete system of algebraically independent differential invariants of the manifold is given by

$$d^\lambda\, \phi^{(\sigma)} \qquad (\lambda = 1,\, 2 \ldots . n\,;\ \sigma = 1,\, 2 \ldots . r)$$

together with the algebraically independent invariants of a system of r-ary forms. These forms are:

(1) A quadratic $\Sigma a_{\alpha\beta} du_\alpha du_\beta$.

(2) $m - r$ quadratics whose coefficients are the fundamental magnitudes of the second order for the manifold.

(3) $m - r$ forms of order q ($q = 3, \ldots . n$) of which the coefficients are the fundamental magnitudes of order q, and which may be expressed in terms of the a's, their derivatives, and the fundamental magnitudes of the second order and their derivatives.

(4) A set of n forms of orders $1,\, 2 \ldots . n$ corresponding to each function ϕ. The coefficients are functions of the a's, the ϕ's, and their derivatives. Forsyth* gives a particular case of this solution, namely that for $n = 3$, $m = 3$, $r = 2$. He, however, assumes at the beginning that there exist certain invariant differential forms corresponding to the sets (2) and (3) whereas in our case these forms have arisen naturally in the course of the development of the method.

§4. Problem (III) still remains for consideration. In the first place, it is clear that $d^\lambda\, \phi^{(\sigma)}$ ($\lambda = 1,\, 2 \ldots . n$; $\sigma = 1,\, 2 \ldots . r$) and the algebraic invariants of the sets (1) and (4) of algebraic forms given in the preceding section are deformation invariants of the quadratic differential form ds^2.†

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*Philosophical Transactions, Ser. A, Vol. 201 (1903), p. 357.

† There is one condition to which the differential form considered is subject. It is assumed possible to express it as the sum of the squares of m perfect differentials. This condition involves no limitation on the form, for it may be proved that any form can be so expressed, provided $m \overline{<} \frac{1}{2} r\,(r + 1)$. See Goursat, Leçons sur l'intégration des Équations aux Dérivées Partielles du premier ordre (1891), p. 11.

In addition to these there are others due to the sets (2) and (3), for it is easy to see that there are relations among the fundamental magnitudes of the second order, the a's and their derivatives,[*] and hence some of the invariants due to these sets must be expressible in terms of the derivatives of the a's alone. The solution of problem (I) shows that these are all zero in that particular case. It is interesting in this connection, to note the six quantities $\Theta_1, \Theta_2, \ldots . \Theta_6$ obtained by Forsyth[†] and originally due to Cayley.[‡] The method hitherto pursued leads us no further in the determination of "Gaussian Invariants," and we must have recourse to another. The problem to be solved is the determination of the invariants of a quadratic form, when there are no associated functions ϕ. Use is made of Lie's general method by the introduction into the group hitherto used of a certain number of new variables which are invariant to the group.[||] Let the quadratic form be

$$\sum_{h=1}^{r} \sum_{k=1}^{r} a_{hk}\, du_h\, du_k,$$

and introduce invariant variables $\alpha_1, \alpha_2, \ldots . \alpha_r$. The α's are functions of the u's, and the u's are taken to be functions of the α's, which are a set of independent variables. We now seek for invariants, under the most general transformation on the u's, which involve as variables

(1) the u's and their derivatives with respect to the α's,
(2) the a's and their derivatives with respect to the u's,
subject to the condition that the quadratic form is invariant.

It is easy to see

(a) that $\sum_{h=1}^{r} \sum_{k=1}^{r} a_{hk} \frac{\partial u_h}{\partial \alpha_\lambda} \frac{\partial u_k}{\partial \alpha_\mu}$ $(\lambda, \mu = 1, 2, \ldots . r)$ is an absolute invariant,

(b) that if H is any absolute invariant so also is $\frac{\partial H}{\partial \alpha}$.

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[*] In the case of a surface in space of three dimensions the Gaussian curvature leads to such a relation.

[†] Philosophical Transactions, Series A, Vol. 202 (1903), p. 306.

[‡] Collected Mathematical Papers, Vol. XII, pp. 12, 13.

[||] Lie, Math. Ann., Vol. XXIV (1884), p. 564, lines 22 et seq.

These two statements (a) and (b) are sufficient to give the clue to the complete set of invariants sought.

Let $T_{\lambda\mu}$ denote the expression (a), and form all possible derivatives up to and including the order $n-1$, of the T's with respect to the α's. These are a complete functionally independent set of invariants. It is easy to prove the independence of this set, for if α_λ is taken to be u_λ, $(\lambda = 1, 2 \ldots\ldots r)$ the T's and their derivatives become the a's and their derivatives, and the a's are arbitrary functions of their arguments. Hence the number of functionally independent invariants obtained is

$$r\,\frac{(r+1)}{2}\,\frac{(n+r-1)!}{(r-1)!\;n!}$$

Now the set of equations for these invariants obtained by the Lie process is precisely that considered by C. N. Haskins,* with certain additional terms due to the presence of derivatives of the u's with respect to the a's. He shows that if $n > 3$ the equations are independent, and therefore it is certain that if $n > 3$ our equations are independent. Now they possess altogether as variables

$$r\,\frac{(r+1)}{2}\,\frac{(n+r-1)!}{(r-1)!\;n!}\ a\text{'s and their derivatives,}$$

$$r\left\{\frac{(n+r)!}{(r-1)!\;(n+1)!}-1\right\}\ \text{derivatives of the } u\text{'s.}$$

Also the equations are in number

$$r\left\{\frac{(n+r)!}{(r-1)!\;(n+1)!}-1\right\},$$

and therefore they possess exactly

$$\frac{r\,(r+1)}{2}\cdot\frac{(n+r-1)!}{(r-1)!\;n!}$$

functionally independent common solutions. But this number has been obtained, and therefore the system of equations has been completely solved. Any invariant is therefore a function of the T's and their derivatives, and the Gaussian Invariants are the particular functions which do not involve the derivatives of the u's with respect to the α's. These may be obtained by processes of elimination, and hence the problem of the determination of the Gaussian Invariants of a quadratic form is completely solved.

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* Trans. Amer. Math. Soc., Vol. 3 (1902), pp. 74-76.

It is clear that the introduction of any functions $\phi\ (u_1, u_2, \ldots\ u_r)$ leads to additional invariants $\frac{d\phi}{d\alpha}$ etc., and therefore the parameters may be obtained in this way. The extension to any number of forms of any degree is immediate, but this question is reserved for future discussion.

BRYN MAWR COLLEGE, PENNA., *April*, 1905.